Hindusthan International Journal of Multidisciplinary Research (HIJOMR)
Volume 1 Issue 3, March 2025, pp.1-6

FACTORIZATION OF k - POSINORMAL OPERATOR ON HILBERT SPACE

Dr.S.Anuradha[1], P.Rekha[2], S.Renuga Devi[3]

[1]Professor & Head, Department of Mathematics,
Hindusthan College of Arts & Science, Coimbatore, Tamil Nadu, India.
anuradha.s@hicas.ac.in
[2,3]Asistant Professor, Department of Mathematics,
Hindusthan College of Arts & Science, Coimbatore, Tamil Nadu, India.
rekha.p@hicas.ac.in, renugadevi.s@hicas.ac.in

ABSTRACT

A mathematical branch of functional analysis that analyzes linear operators on function spaces. The study of linear operators in topological vector spaces, such as Hilbert spaces, and their features is known as operator theory. In this paper, we look at the factorization of k-Posinormal operators on Hilbert space. We expand the Posinormal operator to the k-Posinormal operator, which acts on a Hilbert space, and present some of its theorems as well as the k-Posinormal operator's behavior during factorization.

Keywords: *Posinormal, k-Posinormal, coposinormal, heminormal.*

1. INTRODUCTION

Let H be a complex Hilbert space and B(H) denote the algebra of all bounded linear operators on H. A vector space that has an inner product that generates a function representing distance for which the space is a complete metric space is known as a Hilbert space. A Banach space is a specific instance of a Hilbert space.

A study on Hilbert space operators are very significant in analysis and applications in quantum mechanics. In quantum mechanics the Hilbert space and their properties provide the correct mathematical tool to formalize the law of quantum mechanics and observables of a system are represented by a space of linear operators on a Hilbert space H.

Here, the linear operator as k-posinormal. We explore some theorems using k-Posinormal in factorization.

2. DEFINITION

An operator T in a Hilbert space H is said to be k - Posinormal if $T^k T^{*k} \leq c^2 T^{*k} T^k$ for some c > 0 and k is a natural number.

Using this definition, we proved some theorems in factorization.

2.1 SOME DEFINITIONS

- **Posinormal Operator:**
  An operator A ∈ B(H) is called Posinormal, if $T T^* \leq c^2 T^* T$ for some c > 0 [1].
- **k-Posinormal Operator:**
  An operator A ∈ B(H) is called k-Posinormal, if $T^k T^{*k} \leq c^2 T^{*k} T^k$ for Some c > 0, where k is a positive integer [2].

- **Hyponormal Operator:**
  An operator A ∈ B(H) is called Hyponormal, if $A^{*}A \geq AA^{*}$.[4].
- **M – Parahyponormal Operator:**
  An operator T in a Hilbert space H is M – Parahyponormal, if $\|Tx\|^2 \leq$ M $\|TT^{*}x\|$ for every unit vector x in H.
- **Heminormal Operator:**
  An operator A ∈ B(H) is called Heminormal, if T is hyponormal and T∗T commutes with TT∗.
- **Class A:**
  An operator T belongs to **class A,** if and only if (T∗|T|T)1/2 ≥ T∗T.

we can see from the definitions, as expected,
For p = 1,
(p, k) – Posinormal = k – Posinormal [3]
For k = 1,
k – Posinormal = 1 – Posinormal = Posinormal.

Also we can easily verify that,

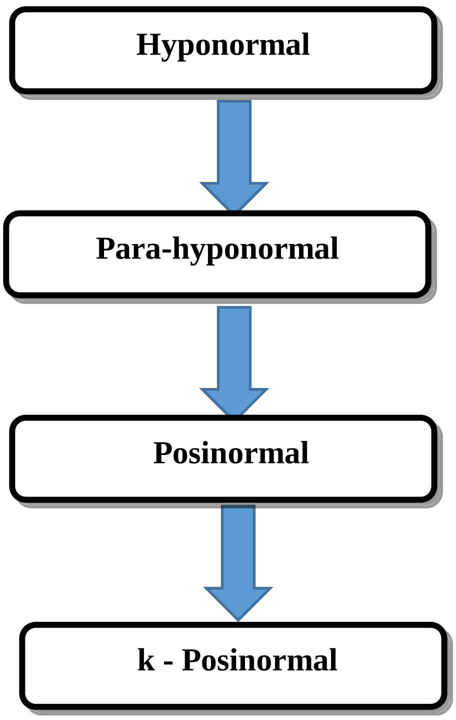

Figure 1

Figure 2

3. K - POSINORMAL OPERATORS ON HILBERT SPACE

3.1 THEOREM

Let A, B and S be operators, then the operator $T = A^{k/2}S$ is a k-posinormal operator if
1. $B \geq A \geq 0$
2. $kSk \leq 1$
3. $B = c^2 S^* A^k S$.

Proof

An operator T is k-posinormal, if $T^k T^{*k} \leq c^2 T^{*k} T^k$.
Suppose that there exist operators A , B and S which satisfies the above three conditions. Then,

$$c^2(A^{k/2}S)^*(A^{k/2}S) - (A^{k/2}S)(A^{k/2}S)^*$$
$$= c^2(S^*A^{k/2}A^{k/2}S) - (A^{k/2}SS^*A^{k/2})$$
$$= c^2 S^* A^k S - A^{k/2} SS^* A^{k/2}$$
$$= B - A^{k/2} SS^* A^{k/2} \qquad \text{[from(3) condition]}$$
$$\geq A - A^{k/2} SS^* A^{k/2}$$
$$= A^{k/2}(I - SS^*)A^{k/2}$$
$$\geq 0$$

Therefore, $c^2(A^{k/2}S)^* (A^{k/2}S) \geq (A^{k/2}S)(A^{k/2}S)^*$
$\Rightarrow$ $A^{k/2}S$ is k-posinormal.

3.2 THEOREM

If T is a k-posinormal operator, then there exist operators A, B and S which satisfy,
1. $B \geq A \geq 0$
2. $\|S\| \leq 1$
3. $B = c^2 S^* A^k S$

then T can be written in the form $T = A^{k/2}S$.

Proof

Since T is a k-posinormal operator,

$$T^k T^{*k} \leq c^2 T^{*k} T^k.$$

Let $T^* = U(TT^*)^{k/2}$ be a polar decomposition of T^*, $A = TT^*$ and $B = c^2 T^*T$.
Then, since T is k-posinormal we have $B \geq A \geq 0$.

Also, $B = c^2 T^*T$
$$= c^2 \; U \; (TT^*)^{k/2} (TT^*)^{k/2} \; U^*$$
$$= c^2 U (TT^*)^k \; U^*$$
$$= c^2 U A^k U$$

If we consider $B = U^*$, Then $kSk \leq 1$, $B = c^2 \; S^* A^k S$ and

$T = (TT^*)^{k/2} U^* = A^{k/2}S.$

3.3 THEOREM

An operator $T = A^{k/2}S$ which satisfies

1. $B \geq A \geq 0$
2. $kSk \leq 1$
3. $B = c^2 S^* A^k S$ is coposinormal, i.e., T^* is posinormal.

Proof

T^* is posinormal. if, $T^*T \leq c^2 TT^*$

$$\begin{aligned} c^2 TT^* - T^*T &= c^2 (A^{k/2}S)(A^{k/2}S)^* - (A^{k/2}S)^* (A^{k/2}S) \\ &= c^2 A^{k/2} S S^* A^{k/2} - S^* A^k S \\ &= c^2 A^{k/2} S S^* A^{k/2} - \frac{B}{c^2} \\ &= c^2 A^{k/2} S S^* A^{k/2} - A^k \quad \text{[from(3) condition]} \\ &= A^{k/2} [c^2 S S^* - I] A^{k/2} \\ &\geq 0 \end{aligned}$$

$\Rightarrow T^*$ is posinormal.

3.4 THEOREM

If A, B, S are normal operator then $T = A^{k/2} S$ which satisfies

1. $B \geq A \geq 0$
2. $kSk \leq 1$
3. $B = c^2 S^* A^k S$ is heminormal.

Proof

A normal operator T is heminormal if T is hyponormal and $T^* T$ commutes with TT^*.
Teishiro Saito [7] have already proved T is hyponormal.

$$\begin{aligned} T^*TTT^* &= (A^{k/2}S)^*(A^{k/2}S)(A^{k/2}S)(A^{k/2}S)^* \\ &= S^* A^{k/2} A^{k/2}S A^{k/2}SS^*A^{k/2} \\ &= S^* A^k S A^{k/2}SS^*A^{k/2} \\ &= \frac{B}{c^2} A^{k/2}SS^*A^{k/2} \quad \text{[from(3) condition]} \end{aligned}$$

$$\begin{aligned} TT^*T^* T &= (A^{k/2}S)(A^{k/2}S)^* (A^{k/2}S)^* (A^{k/2}S) \\ &= A^{k/2} SS^*A^{k/2} S^* A^k S \\ &= A^{k/2} SS^* A^{k/2} \frac{B}{c^2} \quad \text{[From(3) condition]} \end{aligned}$$

$\Rightarrow T^* TT T^* = T T^*T^* T$

$\Rightarrow$ T is heminormal.

3.5 THEOREM

An operator $T = A^{k/2}S$ which satisfies

1. $B \geq A \geq 0$

2. $kSk \leq 1$

3. $B = c^2 S^* A^k S$ is of class A. [8], [9].

Proof

An operator T is of class A if, $(T^*|T|T)^{k/2} \geq T^*T$

$$((A^{k/2}S)^* A^{k/2}S\, A^{k/2}S)^{k/2} - (A^{k/2}S)^* (A^{k/2}S)$$
$$\Leftrightarrow (S^* A^{k/2}(A^{k/2}S)\, S^* A^{k/2} A^{k/2}\, S)^{k/2} - S^* A^K S$$
$$\Leftrightarrow (S^* A^k SS^* AS)^{k/2} - S^* A^k S$$
$$\geq 0$$

$\Rightarrow$ T is of class A.

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AUTHOR PROFILE:

Author – 1

Dr.S.Anuradha, Professor, Head Department of Mathematics, Data Science and Analytics is having 29 Years Teaching and Research Experience. She received Best Teacher Award 2013 in Akshaya Institute of Management studies Coimbatore , Grade A award for Team Leader in Task Based Training (TBT-2019) conducted by e-Yantra Lab Setup Initiative(eLSI), IIT BOMBAY, Vocational Service Award 2021 from Rotary Club of Coimbatore DownTown in 2021, Most Accomplished Spoken Tutorial Educator Resource Award in 2023 by SPOKEN TUTORIAL, IIT BOMBAY. She is a Teacher invitee in 34th Mid Year Meeting of Indian Academy of Sciences in the year 2023. She published 73 International Journal, 12 Books, 9 Book Chapter and 1 patent. She is Editorial Board Member in International Research Conference Series-Scientific and Technical Committee and Editorial board Member on mathematical and Computational Sciences and Reviewer in International Journals of American Journal of Applied Scientific Research, Science Publishing Group, Advances in Applied Research, www.IndianJournals.com,Latin American Applied Research, Applied Mathematics & Information Sciences, Nature Publishing Group. 7 PhD Scholars and 18 MPhil Scholars awarded Degree under her guidance.

Author – 2

Mrs.P.Rekha, designated as Assistant Professor, Department of Mathematics, Hindusthan College of Arts & Science, Coimbatore. She has 12 years of teaching experience. She has Published 6 books and 2 Papers in Conference Proceedings. Her field of interest is Operator Theory.

Author – 3

Mrs. S Renuga Devi , designated as Assistant Professor, Department of Mathematics, Hindusthan College of Arts & Science, Coimbatore. She has 15 years of teaching experience. She has Published 5 books and 1 Paper in International Journal and 2 Papers in Conference Proceedings. Her field of interest is Graph Theory.